Verification and search algorithms for causal DAGs

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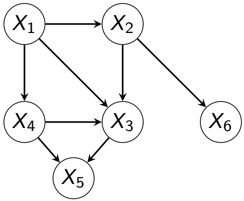
* Equal contribution

Motivation

Underlying data generation process (modelled as a DAG)

G



e.g. $X_4 = f_4(X_1; "_4)$

specific to node X_4

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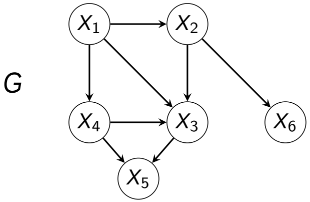

Observational data D

	X_1	X_2	X_3	X_4	X_5	X_6
Sample 1	0.3	0.4	0.1	-0.5	0.2	-0.3
Sample 2	0.1	1.2	0.6	-0.2	-0.1	-0.4
⋮	⋮	⋮	⋮	⋮	⋮	⋮

e.g. $X_4 = f_4(X_1; \varepsilon_4)$

specific to node X_4

Motivation

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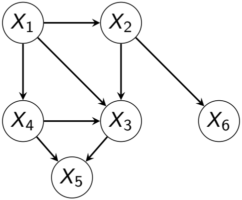

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Motivation

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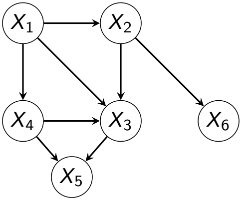

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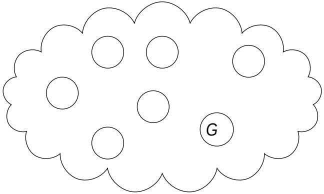
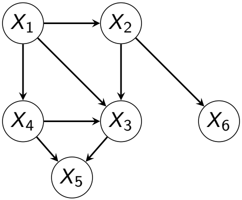
Identify possible graphs
via conditional dependencies
(e.g. PC [SGSH00], GES [Chi02])

Equivalence
class of DAGs



Motivation

G



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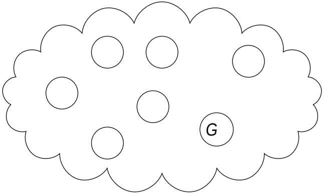
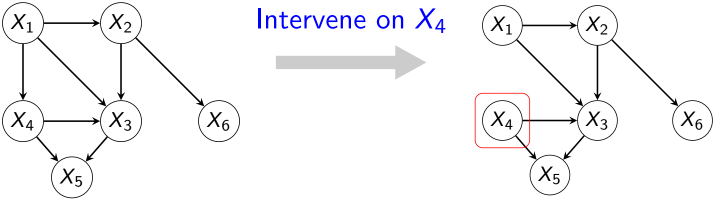


Which is G ?

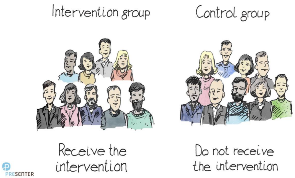
Two ways forward

1. Make model assumptions
   e.g. $X_4 = f_4(X_1; \varepsilon_4) = \quad X_1 + \varepsilon_4$, where ε_4 is non-Gaussian
2. Perform interventions (Our focus)
   e.g. set $X_4 = 0.5$, then draw samples from the resulting intervened causal graph

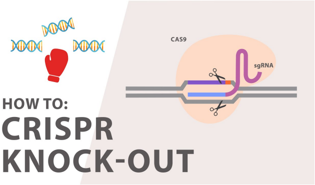


Interventions in real-life

Randomized controlled trials



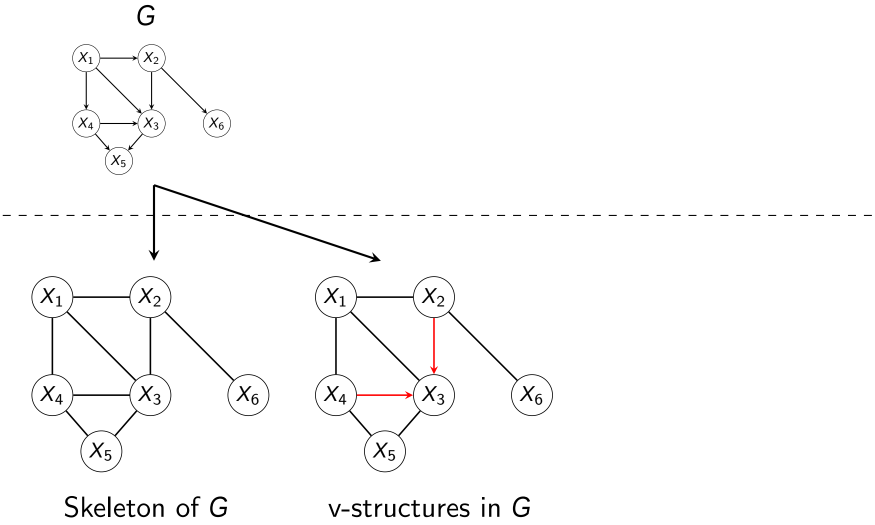
Gene knockout experiments



Can be expensive to perform) Minimize number of interventions!

Image credit: https://i.ytimg.com/vi/vyVyg4Bf2JY/maxresdefault.jpg
https://i.ytimg.com/vi/1BSiFfyObTY/maxresdefault.jpg

What can we learn?



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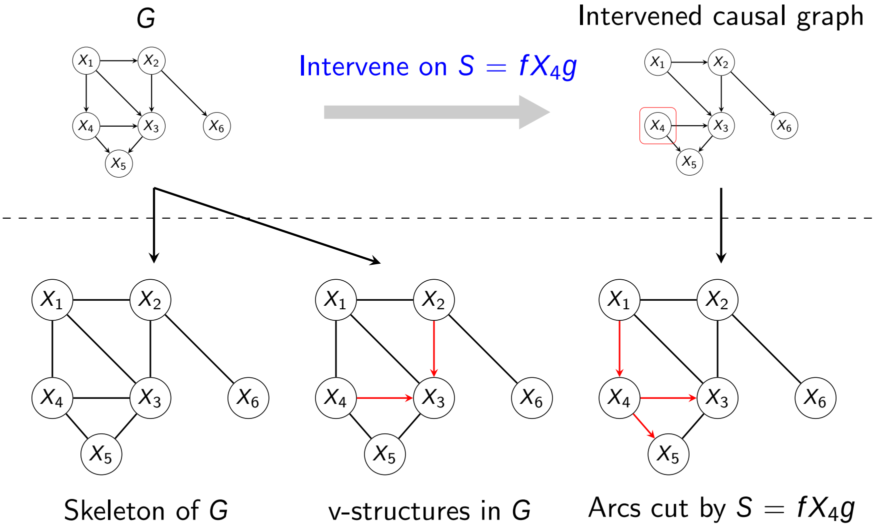
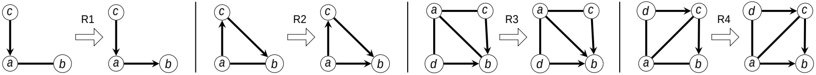


Meek rules

Meek rules [Mee95]:
A set of 4 arc orientation rules that are *sound* and *complete*
(with respect to arc orientations with acyclic completion)



If b a,
then v-structure

If b a,
then cycle

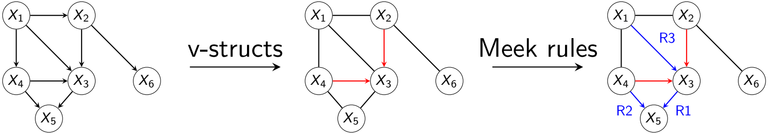
If b a, then the unoriented arcs would have been *oriented in the same way in all DAGs within the equivalence class* (via R2)
(See next slide on essential graphs)

Meek rules converge in polynomial time [WBL21, Algorithm 2].

Essential graph

We can represent an equivalence class with a partially oriented DAG called the *essential graph*

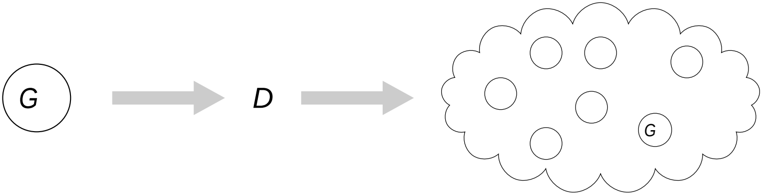
- Orient *u* ! *v* if *all* DAGs agree on this direction
- An unoriented arc if there are two distinct DAGs G_1 and G_2 in the equivalence class orient it differently



G — Essential graph of *G*

Problem setup

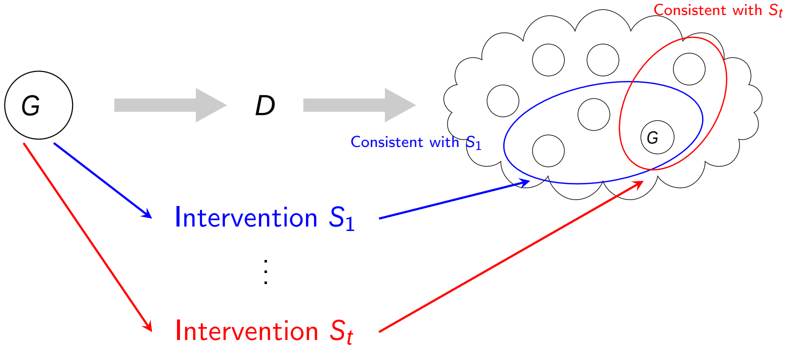
Identify G



Can be represented by
an essential graph
(partially oriented DAG)

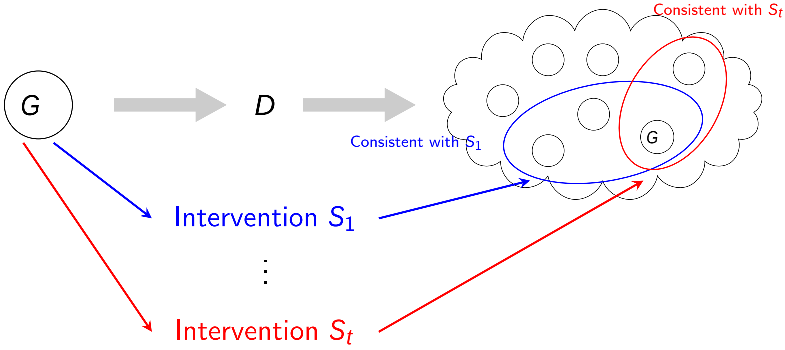
Problem setup (using interventions)

Identify G using as few interventions as possible (minimize t)



Problem setup (using **atomic** interventions)

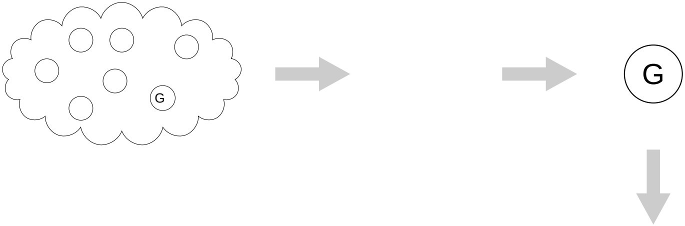
Identify G using as few interventions as possible (minimize t)



Simplifying assumption for this talk:
Each intervention is on a single node, i.e. $|S_1| = \cdots = |S_t| = 1$

Wait a minute... we have domain experts!

Problem solved with zero interventions!



Do stu with
discovered causal graph G

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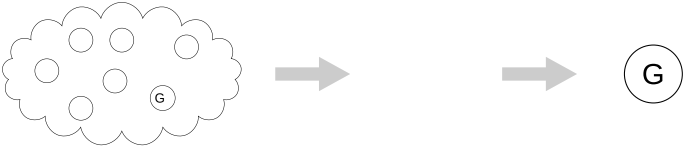

Experts can be wrong...

Do stu with
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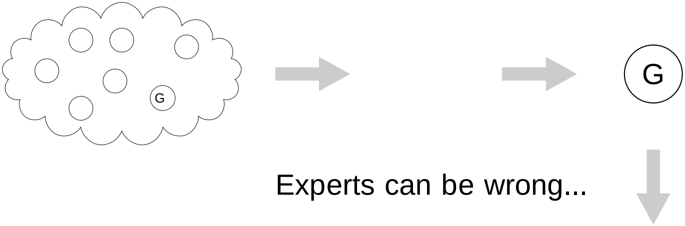
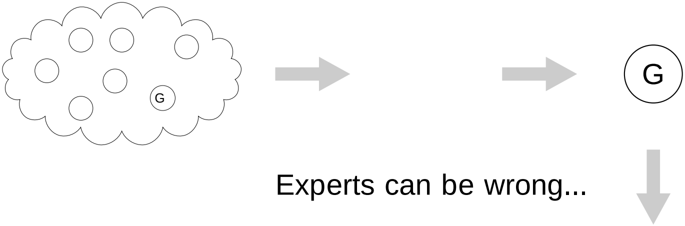


Wait a minute... we have domain experts!

How do we even check $\hat{G} = G$?

~~Problem solved with zero interventions!~~



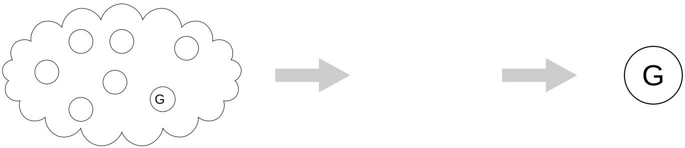
Experts can be wrong...

Image credit:
https://dribbble.com/shots/14489872-Devil
https://dribbble.com/shots/3759014-Atomic-Illustrations/attachments/3759014-Atomic-Illustrations?mode=media
https://img.favpng.com/23/12/11/questionnaire-survey-methodology-png-favpng-CW1Hb5zY6b47rPbAnvWgwEHPK.jpg

The veri cation problem

Goal: Determine if G = G

(G) = minimum number of interventions to answer $G \stackrel{?}{=} G$



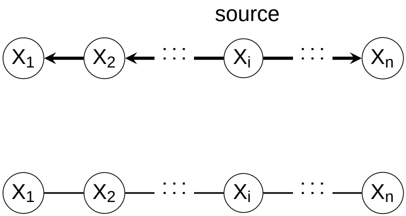
We know: Intervening on v orients all arcs incident to v

Trivial solution: Compute minimum vertex cover (MVC) on unoriented arcs! i.e. (G) MVC(unoriented)

Verification: Problem with trivial solution

G

Essential graph



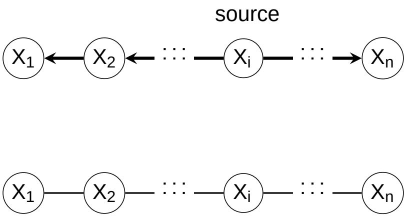
Any line graph with a single source $i \in \{1, \dots, n\}$ has the same essential graph above of an unoriented line

MVC(unoriented in essential graph) = $\frac{n}{2}$

Verification: Problem with trivial solution

G

Essential graph



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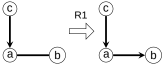
MVC(unoriented in essential graph) $= \frac{n}{2}$

Optimal: Just 1 intervention needed!

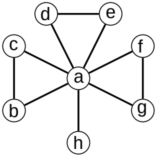
- Intervene on $X_i \Rightarrow$ Orient $X_{i-1} \leftarrow X_i$ and $X_i \rightarrow X_{i+1}$
- Apply Meek R1 to orient the rest
- If not fully oriented or $\exists$ disagreeing arcs, then $\widehat{G} \neq G$



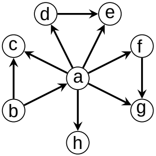
Veri cation: What was known

(Simplifying assumption: single undirected connected component)

1. [SMG+ 20]: (G) j $\frac{!\ (G)}{2}$ k
2. [PSS22]: $\frac{n \quad r}{2}$ (G) n r (Note: 2-apx gap)



Essential graph



One possible DAG

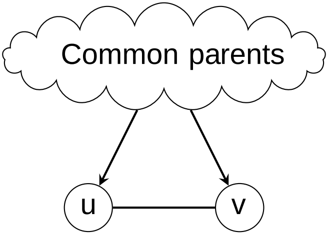
n = 8 nodes; r = 4 maximal cliques; largest clique (G) = 3

[SMG+ 20]: (G) 1; [PSS22]: 2 (G) 4

Can we do better?

Verification: A complete characterization via covered ed

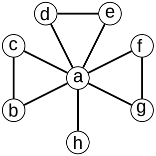
- Meek rules $\Rightarrow$ Outperform MVC (unoriented)
- Surprisingly, enough to compute MVC on a subset of edges
- Covered edges [Chi95]:
  $u \to v$ is covered edge $\iff$ $Pa(u) \setminus \{v\} = Pa(v) \setminus \{u\}$



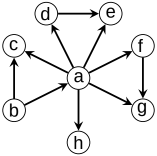
Claim: Necessary and sufficient to intervene on MVC (covered)

Proof: Simple (but subtle) using the notion of covered edges

Veri cation: Comparing to prior work



Essential graph

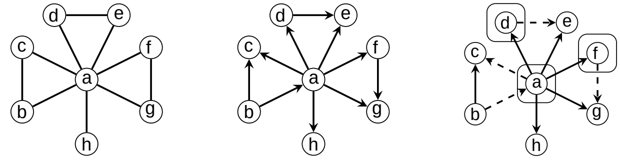


DAG

! (G) = 3, n = 8, r = 4) (G) 1; 2 (G) 4

Can we do better?

Veri cation: Comparing to prior work



Essential graph DAG Covered edges

! (G) = 3, n = 8, r = 4) (G) 1; 2 (G) 4

Can we do better? Yes

We get exact (G) for each G in the equivalence class

In fact, every DAG in this equivalence class needs 3 or 4 interventions, so the existing bounds on (G) are not tight.

Veri cation: E cient computation

In general, MVC is NP-hard and we can only get a 2-approximation in polynomial time... [PSS22] also has a 2-approximation to (G).

Image credit: https://wronghands1.com/2018/01/12/parts-of-the-elephant-in-the-room/

Veri cation: E cient computation

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Claim: Covered edges form a forest.
Implication: MVC(covered) can be computed exactly in linear time.

Image credit: https://wronghands1.com/2018/01/12/parts-of-the-elephant-in-the-room/

Easy re-interpretation of known facts via covered edg

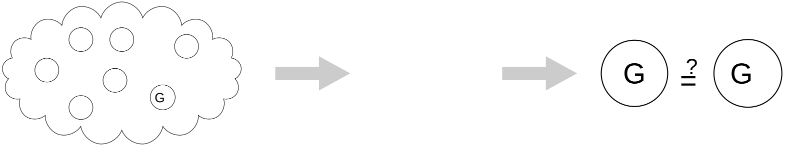
- Covered edges of clique K_n: $v_1 \to v_2, \ldots, v_{n-1} \to v_n$
- Covered edges of a tree: incident edges to root vertex
- Necessity of separating system for non-adaptive interventions
  - [Chi95]: Two graphs are equivalent $\iff$ there is a sequence of covered edge reversals to transform between them.
  - Unoriented edge $\Rightarrow$ Covered edge for *some* DAG in eq. class.
  - Conclusion: any *non-adaptive* search must cut *all* edges.
- Covered edge *cannot* have both endpoints as a sink of any maximal clique $\Rightarrow$ $(G) \quad n \quad r$ (result of [PSS22]).

(Slide catering to domain experts. If interested, pause to read; Else, skip)

The veri cation problem X

Can determine $G \stackrel{?}{=} G$

- Using (G) = MVC(covered) interventions
- Computable in polynomial time



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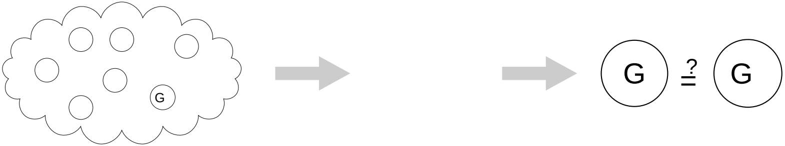
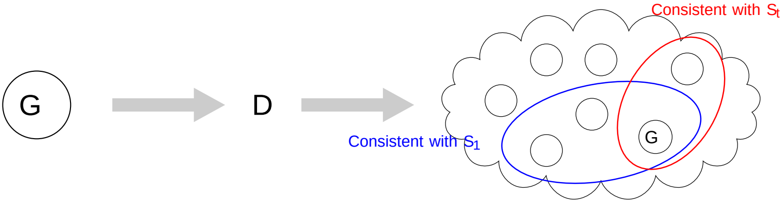


What about actually searching for G without the expert?

The adaptive search problem

Goal: Identify G using as few interventions as possible



We know that at least (G) interventions isnecessary

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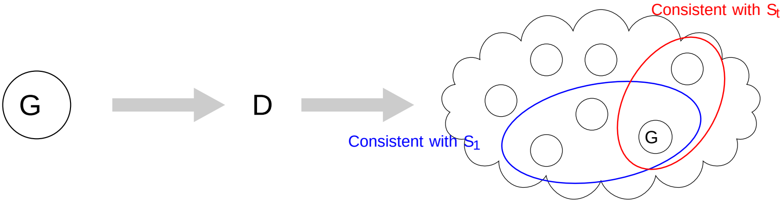


We know that at least (G) interventions isnecessary

Punchline: O(log n (G)) interventions su ce

\Search is almost as easy as veri cation"

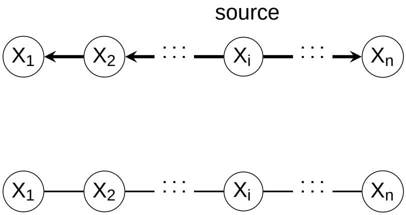
Adaptive search $O(\log n \quad (G))$ interventions su ce

Prior works only have theoretical guarantees for special classes of graphs: cliques, trees, intersection incomparable graphs, :::

$(\log n \quad (G))$ interventions are necessary in the worst case

G

Essential graph



$(G) = 1$ and identifying G is equivalent to binary search

Covered edges are $X_{i-1} \quad X_i\ !\ X_{i+1}$) Need to \hit" X_i

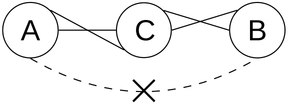
Each intervention orients \one side" via Meek rule R1

Adaptive search: How it works

[HB12]: Intervene and remove oriented arcs $\Rightarrow$ Chordal graph

[GRE84]: For any chordal graph, there exists a clique separator C such that

- $|A|, |B| \le n/2$
- C is a clique, i.e. $|C| \le \omega(G)$
- Computable in polynomial time



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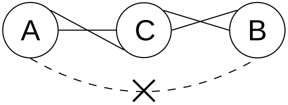


Algorithm: Find clique separator in each component; Intervene on each node in all clique separators; Recurse

Analysis:

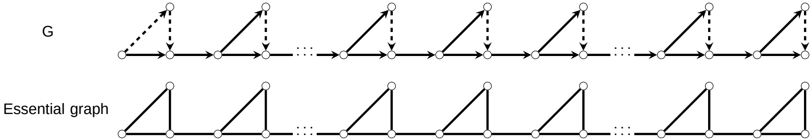
- $O(\log n)$ rounds suffice [GRE84]
- $O(\omega(G))$ per round We prove new lower bound on $\omega(G)$

A stronger (but not computable) lower bound

Intuition [HB14]: In any interventional essential graph, interventions across di erent \connected components" do not help.

Claim: Fix an essential graph and some DAG G in it. Then,

$$(G) \qquad \underset{\substack{\text{connected components} \\ H\ 2\ \text{after removing oriented arcs}}}{X} \qquad \frac{!\ (H)}{2}$$

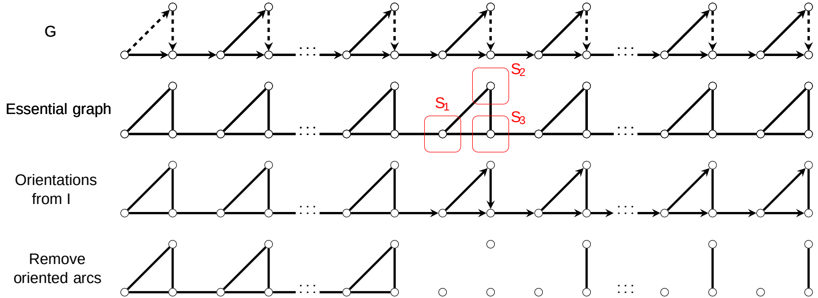


A stronger (but not computable) lower bound

Intuition [HB14]: In any interventional essential graph, interventions across di erent \connected components" do not help.

Claim: Fix an essential graph and some DAG G in it. Then,

(G) max atomic interventions $S_1;\dots;S_t$ X H 2 connected components after removing oriented arcs after interventions $S_1; : : : ; S_t$ $\frac{!\ (H)}{2}$



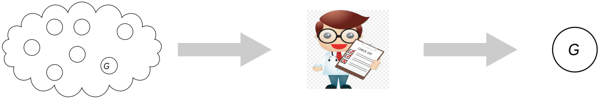
Experiments (Atomic search comparision)

Qualitatively, our algorithm is competitive with the state-of-the-art search algorithms while being 10x faster in some experiments.

Implementation: https://github.com/cxjdavin/verification-and-search-algorithms-for-causal-DAGs

Summary

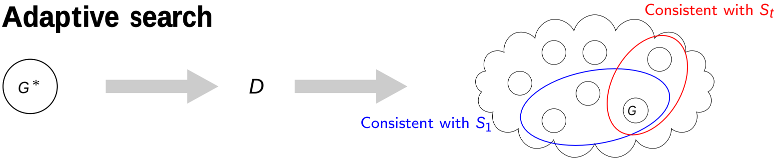
1. Veri cation



Polynomial time *exact characterization* of (G)

$(G) = \texttt{MVC}(\text{covered})$ to determine if $G \stackrel{?}{=} G$

2. Adaptive search



Polynomial time adaptive search algorithm using interventions

$O(\log n \quad (G \quad))$ suffice for *any general graph*

$\Omega(\log n \quad (G \quad))$ worst case necessary

Natural follow up questions

In this work, we studied *veri cation* and *search* under an idealized setting with hard interventions and infinite samples.

Soft interventions may be more realistic in certain real-life scenarios (e.g. effects from parental vertices are not completely removed but only altered); see [KJSB19]

Sample complexities also play a crucial role when one has limited experimental budget; see [ABDK18]

We also make standard assumptions such as the Markov assumption, the faithfulness assumption, and causal sufficiency [SGSH00]. Can we remove/weaken these assumptions?

References I

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